Announcements

1) Ford Recruiting Day
9/22 (Thursday)
12-3 IANS

2) Math talks on tilings
3-4 2046 CB (technical)
7-8:30 1030 CB (general)
(Friday 9/23)

3) Turn in problems 6, 11
in class on Thursday

4) Make-up office hours today 4.30 – 5

More on Horizontal Asymptotes

For $\dfrac{\text{polynomial}}{\text{polynomial}}$

only the highest powers matter!

For, example

Example 1:

$$\lim_{x \to -\infty} \frac{9x^4 - 11x^2 + 2x - \frac{\pi}{2}}{-\sqrt{2}\,x^4 + x^3 - 1{,}022}$$

Same as

$$\lim_{x \to -\infty} \frac{9\cancel{x^4}}{-\sqrt{2}\,\cancel{x^4}} = \boxed{-\frac{9}{\sqrt{2}}}$$

Only highest powers matter!

What about

$$\lim_{x\to\infty} \frac{7x^3+11}{19x^{101}+2x-e} ?$$

this is equal to

$$\lim_{x\to\infty} \frac{7x^3}{19x^{101}}$$

$$= \lim_{x\to\infty} \frac{7}{19} \frac{1}{x^{98}}$$

$$= \frac{7}{19} \lim_{x\to\infty} \frac{1}{x^{98}}$$

$$= \frac{7}{19} \cdot 0 = \boxed{0}$$

Finally,

$$\lim_{x \to -\infty} \frac{10x^{72} + x^5 - 2x^2 + 1}{124x^{69.2} + 5x^3 - 3x^{70}}$$

Same as

$$\lim_{x \to -\infty} \frac{10x^{72}}{-3x^{70}}$$

$$= \lim_{x \to -\infty} -\frac{10}{3} x^2$$

$$= -\frac{10}{3} \lim_{x \to -\infty} x^2$$

$$= -\frac{10}{3} \cdot \infty = \boxed{-\infty}$$

General Rules

You have $\frac{p(x)}{q(x)}$ where

p, q are polynomials.

1) If $\deg(q) > \deg(p)$,

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = \lim_{x \to -\infty} \frac{p(x)}{q(x)} = 0$$

2) If $\deg(q) < \deg(p)$,

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} \text{ and } \lim_{x \to -\infty} \frac{p(x)}{q(x)}$$

are either ∞ or $-\infty$

(check)

3) If $\deg(p) = \deg(q)$,

then $\lim\limits_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x \to -\infty} \dfrac{p(x)}{q(x)}$

= the ratio of the coefficients of the highest power

Example 2: Find

$$\lim_{x \to \infty} \left(\sqrt{x+2} - \sqrt{x} \right)$$

We think the answer could be either 0 or $\sqrt{2}$.

$\infty - \infty$ is most of the time like $\frac{0}{0}$ = more work

We multiply again by the conjugate of

$\sqrt{x+2} - \sqrt{x}$.

$$\frac{\left(\sqrt{x+2}-\sqrt{x}\right)\cdot\left(\sqrt{x+2}+\sqrt{x}\right)}{\left(\sqrt{x+2}+\sqrt{x}\right)}$$

$$=\frac{\left(\sqrt{x+2}\right)^2-\cancel{\sqrt{x}\cdot\sqrt{x+2}}+\cancel{\sqrt{x}\cdot\sqrt{x+2}}-\left(\sqrt{x}\right)^2}{\sqrt{x+2}+\sqrt{x}}$$

$$=\frac{\cancel{x}+2-\cancel{x}}{\sqrt{x+2}+\sqrt{x}}$$

$$=\frac{2}{\sqrt{x+2}+\sqrt{x}}$$

So

$$\lim_{x \to \infty} \left(\sqrt{x+2} - \sqrt{x} \right)$$

$$= \lim_{x \to \infty} \left(\frac{2}{\sqrt{x+2} + \sqrt{x}} \right)$$

$$= \frac{2}{\infty} = \boxed{0}$$

Example 3: Find all horizontal asymptotes for

$$f(x) = \sqrt{x^2+1} - \sqrt{x^2+10x-7}$$

Note that the domain of $\sqrt{x^2+10x-7}$ is not all real numbers, but as we increase our numbers (in absolute value), we end up in the domain

$$\sqrt{x^2+1} - \sqrt{x^2+10x-7}$$

multiply by the conjugate

$$\left(\sqrt{x^2+1} - \sqrt{x^2+10x-7}\right) \cdot \frac{\left(\sqrt{x^2+1} + \sqrt{x^2+10x-7}\right)}{\left(\sqrt{x^2+1} + \sqrt{x^2+10x-7}\right)}$$

$$= \frac{(\cancel{x^2}+1) - (\cancel{x^2}+10x-7)}{\sqrt{x^2+1} + \sqrt{x^2+10x-7}}$$

$$= \frac{-10x+8}{\sqrt{x^2+1} + \sqrt{x^2+10x-7}}$$

$$\lim_{x\to\infty} \frac{-10x+8}{\sqrt{x^2+1}+\sqrt{x^2+10x-7}}$$

$$= \lim_{x\to\infty} \frac{-10x}{\sqrt{x^2}+\sqrt{x^2}}$$

(only highest powers matter)

$$= \lim_{x\to\infty} \frac{-10x}{x+x} = \lim_{x\to\infty} \frac{-10\cancel{x}}{2\cancel{x}}$$

$$= \boxed{-5}$$

We also have to check

$\lim_{x\to-\infty}$ . . .

$$\lim_{x\to-\infty} \frac{-10x+8}{\sqrt{x^2+1}+\sqrt{x^2+10x-7}}$$

$$= \lim_{x\to-\infty} \frac{-10x}{\sqrt{x^2}+\sqrt{x^2}}$$

$\sqrt{x^2} = |x|$ when $x<0$

$$= \lim_{x\to-\infty} \frac{-10x}{|x|+|x|} = \lim_{x\to-\infty} -\frac{10}{2}\cdot\frac{x}{|x|}$$

(Since $x\to-\infty$ means $x<0$)

$$= -5 \lim_{x\to-\infty} \frac{x}{|x|}$$

$$= -5\,(-1) = \boxed{5}$$

Horizontal Asymptotes

at $\boxed{y = \pm 5}$

Caution! It is correct to write

$$\lim_{x \to \infty} \frac{7x+9}{8x-2} = \lim_{x \to \infty} \frac{7x}{8x}.$$

It is **NOT** correct to write

$$\cancel{\frac{7x+9}{8x-2} = \frac{7x}{8x}}$$

The Squeeze Theorem:

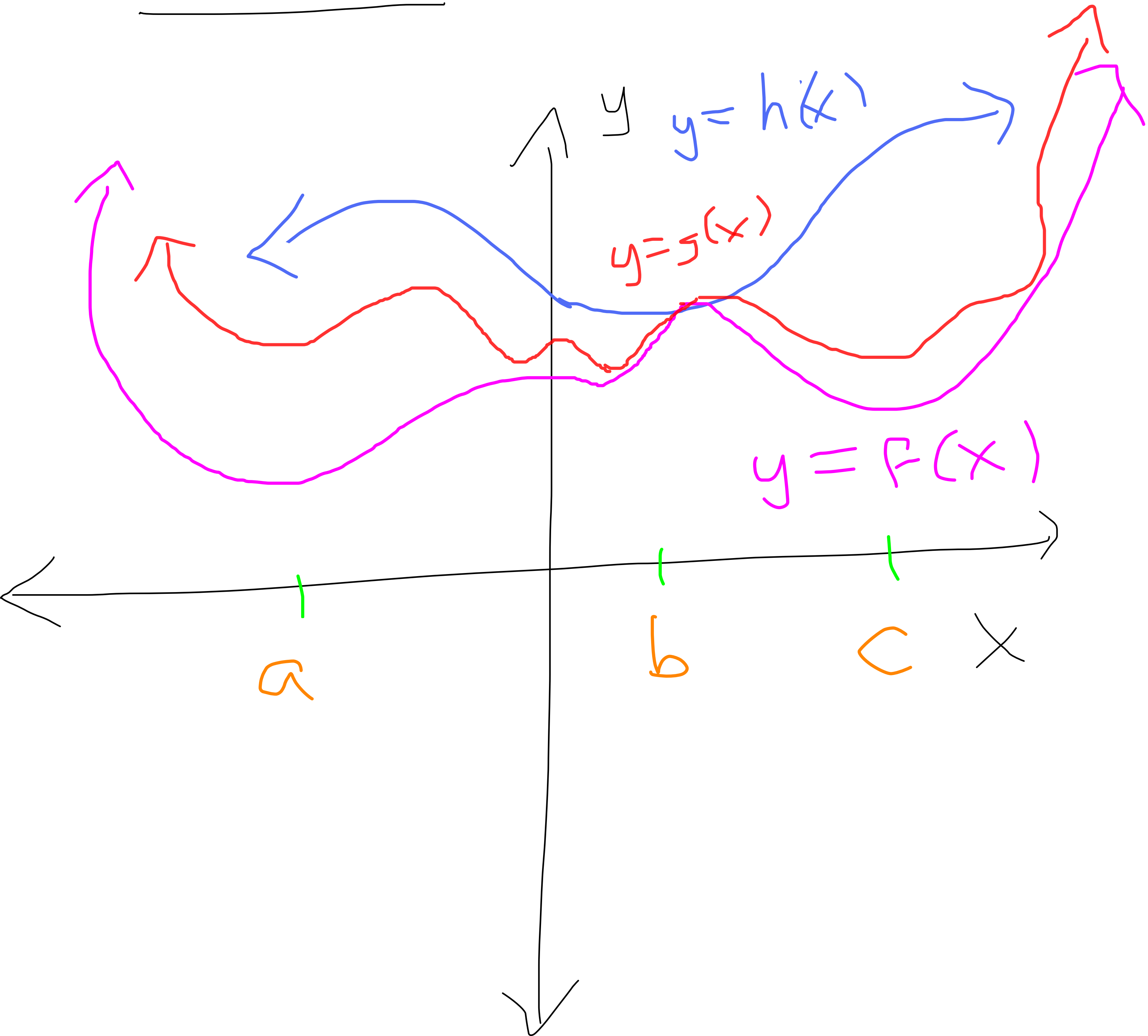
Suppose $b < a < c$ and on (b, c), $f(x) \le g(x) \le h(x)$.

If $\lim_{x \to a} f(x) = L$ and

$\lim_{x \to a} h(x) = L$, then

$$\lim_{x \to a} g(x) = L.$$

Picture:



This also works

for one sided limits

(i.e. $f(x) \le g(x) \le h(x)$ on $[a,b)$

and you want $\lim_{x \to a^+}$)

and also for limits to

$\pm\infty$ (i.e. $f(x) \le g(x) \le h(x)$

on $(-\infty, c)$ and you want

$\lim_{x \to -\infty}$)

Example 4:

Find $\lim_{x \to 2^+} \left((x-2)\sin\left(\frac{1}{x-2}\right) \right)$

Idea: in the picture,

$$(x-2)\sin\left(\frac{1}{x-2}\right) = g(x)$$

We want to find $f(x)$ and $h(x)$ with $\lim_{x \to 2} f(x) = \lim_{x \to 2} h(x)$

and $f(x) \leq g(x) \leq h(x)$

Use the fact that

$$\boxed{-1 \leq \sin(\theta) \leq 1}$$

no matter what θ is!

In particular, if $\theta = \dfrac{1}{x-2}$,

$$-1 \leq \sin\left(\frac{1}{x-2}\right) \leq 1$$

multiply both sides by $x-2$

$$\underbrace{-(x-2)}_{g(x)} \leq (x-2)\sin\left(\frac{1}{x-2}\right) \leq \underbrace{x-2}_{h(x)}$$

$$\lim_{x \to 2} h(x) = \lim_{x \to 2} f(x) = 0,$$

so $\lim_{x \to 2} g(x) = \boxed{0}$

$$\left(g(x) = (x-2)\sin\left(\frac{1}{x-2}\right) \right)$$

Idea of squeeze theorem:

given $\lim_{x \to a} f(x)$.

Isolate a bounded factor of $f(x)$

(i.e $\sin(x)$, $\cos(x)$, $\arctan(x)$)

$$-1 \leq \sin(x) \leq 1$$
$$-1 \leq \cos(x) \leq 1$$
$$-\pi/2 \leq \arctan(x) \leq \pi/2$$

Manipulate the bounds until you see f(x) in the middle.

Our problem was

$$\lim_{x \to 0} (x-2)\sin\left(\frac{1}{x-2}\right)$$

Want to see

$$\star \le (x-2)\sin\left(\frac{1}{x-2}\right) \le \star$$

Another example:

$$\lim_{x \to 3} \left((x-3)^2 \cos\left((x-3)^{1/3} \right) \right)$$

is not a squeeze theorem problem, since you can just plug in $x=3$.

But if we change it to

$$\lim_{x \to 3} \left((x-3)^2 \cos\left(\frac{1}{(x-3)^{1/3}} \right) \right)$$

We know $\boxed{-1 \leq \cos(\theta) \leq 1}$ for all values of θ.

So, with $\theta = \frac{1}{(x-3)^{1/3}}$,

$$-1 \leq \cos\left(\frac{1}{(x-3)^{1/3}}\right) \leq 1$$

Multiply across the inequality by $(x-3)^2$:

$$-(x-3)^2 \leq (x-3)^2 \cos\left(\frac{1}{(x-3)^{1/3}}\right)$$

$$\leq (x-3)^2$$

Now that we've achieved the function we started out with in the middle of the inequality, take limits.

$$\lim_{x \to 3} -(x-3)^2 \leq \boxed{\lim_{x \to 3} (x-3)^2 \cos\left(\frac{1}{(x-3)^{1/3}}\right)}$$

$$\leq \lim_{x \to 3} (x-3)^2$$

$$\lim_{x \to 3} -(x-3)^2 = 0 \quad \text{and}$$

$$\lim_{x \to 3} (x-3)^2 = 0$$

so $0 \le \lim_{x \to 3} (x-3)^2 \cos\left(\frac{1}{(x-3)^{1/3}}\right)$

≤ 0

By Squeeze Theorem,

$$\boxed{\lim_{x \to 3} (x-3)^2 \cos\left(\frac{1}{(x-3)^{1/3}}\right) = 0}$$